TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 170, August 1972

REMARKS ON THE WAVE FRONT OF A DISTRIBUTION[1]

BY

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ABSTRACT. Basic facts about composition and multiplication of distributions as given in [1] are proved using the formulas for the wave front set of the image and pullback of distributions.

1. Let X be a manifold, $D'(X)$ the distributions on X. If a smooth density, dx, is given on X, then any smooth function on X, g, gives rise to a smooth distribution

$$\langle f, g\rangle = \int fg\, dx.$$

These smooth distributions can be pulled back along differentiable maps ϕ: $Y \to X$ (that is, $g \in D'(X)$ goes to $\phi^* g \in D'(Y)$ by the usual pullback of functions). We would like to define a "continuous" extension of this and other notions, so as to include a wider subset of $D'(X)$. This was done in [1, Chapter 2.5], and here we only give a unified treatment with some slight extensions.

According to [1] the question of when a distribution A can be pulled back is answered in terms of a set, $WF(A) \subset T^*(X)$, and a formula for $WF(\phi^* A)$ is obtained when $\phi^* A$ is defined. If $\phi: X \to Z$ is a differentiable map and A satisfies obvious conditions on its support relative to A then $\phi_* A$ can be defined as a distribution on Z. We derive a formula relating $WF(\phi_* A)$ to $WF(A)$. The various theorems of [1] on multiplication and composition of distribution then follow from standard functional constructions.

Definition. If $A \in D'(X)$, $(x_0, k_0) \in T^*(X)$, $k_0 \neq 0$, then $(x_0, k_0) \notin WF(A)$ (wave front of A) if there exists a C^∞ function ϕ with compact support, $\phi(x_0) \neq 0$, and an open cone $\Gamma \subset T^*_{x_0}(X)$ containing k_0, such that for each $k \in \Gamma$ and some C^∞ function ψ with $d\psi(x_0) = k$,

$$t^m |A(\phi e^{-it\psi})| \to 0 \qquad (t \to \infty).$$

This definition is geometric (with no reference to local coordinates in X), and is

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Received by the editors August 10, 1971.

AMS 1969 subject classifications. Primary 4640; Secondary 3520.

Key words and phrases. Distribution, wave front set.

(1) These results were obtained during participation in a joint seminar held by the mathematics department of the Weizmann Institute and the physics department of the University of Tel-Aviv. One of the topics of the seminar was [1]. I would like to thank Professors B. Kostant, B. Simon and S. Sternberg for helpful comments.

easily seen to be equivalent to the one given in [1] via Proposition 2.5.5 and the independence of coordinates given by Theorem 2.5.11′ thereof.

In terms of local coordinates, let $U \subset X$, $x_0 \in U$, be identified with an open subset of $V = \mathbf{R}^n$. Then the tangent space at each point of U is identified with V, and the cotangent space with V^*, so that the cotangent bundle is $U \times V^*$. For a distribution u on X, with compact support in U, we have the Fourier transform, $\hat{u}$, given by

$$\hat{u}(k) = \langle e^{-i(k,x)}, u\rangle \quad \text{for } k \in V^*.$$

With this notation, $(x_0, k_0) \notin WF(A)$ for A as above if there is $\phi \in C_0^\infty$, $\phi(x_0) \neq 0$, an open cone $\Gamma \subset V^*$ and a constant C_m for each m, such that

$$|\widehat{\phi A}(k)| \leq C_m(1 + |k|)^{-m} \quad \text{for all } k \in \Gamma,\ |k| \geq C.$$

For distributions with support in U we define convergence as follows: If Γ is a closed cone in $T^*(U)$, define

$$D'_\Gamma(u) = \{A \in D'(U) \colon WF(A) \subset \Gamma\}.$$

A sequence $A_j \in D'_\Gamma(U)$ *converges* to $A \in D'_\Gamma(U)$ if

(i) $A_j \to A$ in $D'(U)$ (weakly),

(ii) $|k|^m\, |\hat{A}_j(k) - \hat{A}(k)| \to 0$, $|k| \to \infty$, uniformly for k in a closed cone disjoint of Γ.

For a general manifold X we define convergence in $D'_\Gamma(X)$ by partition of unity. A definition utilizing pseudodifferential operators can be found in [1] together with a proof that $C^\infty(X)$ is sequentially dense in $D'_\Gamma(X)$.

2. The following theorem is proved directly in [1, Theorem 2.5.11′]:

Theorem 1. *Let X, Y be manifolds, $\phi\colon Y \to X$ a C^∞ map, and denote the set of normals of the map—*

$$N_\phi = \{(\phi(y), \xi) \in T^*(X) \colon \langle \eta, \xi\rangle = 0 \ \textit{for all}\ \eta \in \phi_* T_y\}.$$

If $A \in D'(X)$ and $WF(A) \cap N_\phi = \emptyset$, we can define the pullback ϕ^*A in one and only one way so that it is equal to the composition $f \circ \phi$ when $A = f$ smooth, and is sequentially continuous from $D'_\Gamma(X)$ to $D'(Y)$ for any closed cone $\Gamma \subset T^*(X)\setminus 0$ with $\Gamma \cap N_\phi = \emptyset$. Moreover,

$$WF(\phi^*A) \subset \phi^* WF(A) = \{(y, \phi_y^*(\xi)) \colon (\phi(y), \xi) \in WF(A)\}.$$

Let $\phi\colon X \to Y$ be a C^∞ map, $A \in D'(X)$, and suppose that either

(a) ϕ is proper ($\phi^{-1}(K)$ compact for any $K \subset Y$ compact) or

(b) A has compact support.

Then $\phi_* A$ is well defined as a distribution on Y by the formula

$$\langle f, \phi_* A\rangle = \langle \phi^* f, A\rangle.$$

For any $S \subset T^*(X)$ we define $\phi_* S \subset T^*(Y)$ by

$$\phi_* S = \bigcup_x (\phi_x^*)^{-1}(S_x) \quad \text{where } S_x = S \cap T_x^*(X).$$

If S is compact, so is $\phi_* S$.

Also, given a pair of mappings $\phi_1 : X \to Y$, $\phi_2 : Y \to Z$,

$$(\phi_2 \circ \phi_1)_*(S) = \phi_{1*}(\phi_{2*} S).$$

Theorem 2. *$WF(\phi_* A) \subset \phi_* WF(A)$ for $A \in D(X)$ as above.*

Proof. Every map can be factored as $\phi = \pi \circ \iota$, where $\iota : X \to X \times Y$ is $x \mapsto (x, \phi(x))$, and $\pi : X \times Y \to Y$ is projection onto the second factor. It therefore suffices to prove the theorem for the cases where $\phi = \iota$ is an imbedding and $\phi = \pi$ is projection as above.

In the case that $\iota : X \to Y$ is an imbedding, we can make the more refined assertion that

$$WF(\iota_* A) = \iota_* WF(A).$$

The WF is independent of coordinates, and is defined locally so we can apply a partition of unity and change of coordinates. Therefore, it suffices to prove this in the case where $X \subset \mathbf{R}^m$, $Y = X \times Z$, $Z \subset \mathbf{R}^n$, and ι defined by $\iota(x) = (x, 0)$. In this case, for any $S \subset T^*(X)$,

$$\iota_*(S) = S \times T_0^*(Z), \qquad T^*(X \times Z) = T^*(X) \times T^*(Z), \qquad \iota_*(A) = A \otimes \delta.$$

This gives immediately $WF(\iota_* A) = \iota_* WF(A)$ by the next (independent) theorem on the wave front of a tensor-product (for $\subset$) and direct observation (for $\supset$).

Now we examine the case $\phi = \pi : X \times Y \to Y$. Here $\phi_x^*(l) = (0, l)_{(x,y)}$ for any $l \in T_y^*$. Thus $l \in \phi_*(S)$ iff $\exists x$ with $(0, l)_{(x,y)} \in S$. Suppose that $l_0 \notin \phi_*(WF(A))$. Then for each x, $(0, l_0)_{(x,y)} \notin WF(A)$. By assumption the possible $(x, y) \in \phi^{-1}(y)$ form a compact set. Each (x, y) has a coordinate neighborhood and ψ such that $\langle e^{-i(l,y)}, \psi A\rangle$ vanishes infinitely rapidly at infinity for all l in a conical neighborhood of l_0. By compactness, a finite number of cones will do. By applying a partition of unity w.r.t. to x we are reduced to the case where A is of compact support, and the result follows from the definition.

Theorem 3. *If $A \in D'(X)$, $B \in D'(Y)$ with $\Gamma_1 = WF(A) \subset T^*(X)$, $\Gamma_2 = WF(B) \subset T^*(Y)$, then*

$$WF(A \otimes B) \subset \Gamma_1 \times \Gamma_2 \cup \Gamma_1 \times 0_y \cup 0_x \times \Gamma_2$$

(*where* $0_x = \{(x, 0)\} \subset T^*(X)$ *the zero section*).

The proof is direct from the definitions.

Note that these three operations (ϕ^*, ϕ_* and $\otimes$), as well as compositions of them, are sequentially continuous. This will be a useful tool in extending other operations in a sequentially continuous mode.

3.

Theorem [1, **Theorem** 2.5.10]. *Let* Γ_1, Γ_2 *be two closed cones in* $T^*(X)\setminus 0$ *satisfying* $\Gamma_1 + \Gamma_2 \subset T^*(X)\setminus 0$ (*where* $\Gamma_1 + \Gamma_2 = \{(x, \xi_1 + \xi_2) : (x, \xi_j) \in \Gamma_j\}$). *Then the product* A_1A_2 *of* $A_j \in D_j'(X)$ *can be defined* (*in a unique way*) *and*

$$WF(A_1A_2) \subset \Gamma_1 + \Gamma_2 \cup \Gamma_1 \cup \Gamma_2.$$

Proof. For functions $f_1, f_2 \in D(X)$ the product is

$$f_1(x) \cdot f_2(x) = \Delta^*(f_1 \otimes f_2)$$

where $f_1 \otimes f_2$ is in $D(X \times X)$, $f_1 \otimes f_2(x, y) = f_1(x)f_2(y)$, and $\Delta : X \to X \times X$ is the diagonal map $\Delta : x \to (x, x)$ with $\Delta^*g(x) = g(x, x)$ for any $g(x, y) \in D(X, Y)$.

In view of our previous remark, a multiplication in $D'(X)$ must also satisfy $A_1A_2 = \Delta^*(A_1 \otimes A_2)$ whenever it is defined.

The only restriction on the above construction is on applying Δ^*.

For any distribution $A \in D'(X \times X)$ and the diagonal map Δ the restriction is $WF(A) \cap N_\Delta = \varnothing$ where $N_\Delta = \{(x, x, \eta, -\eta) : \eta \in T^*(X)\}$.

In our case, $A_j \in D'_{\Gamma_j}(X)$, so

$$WF(A_1 \otimes A_2) \subset \Gamma_1 \times \Gamma_2 \cup \Gamma_1 \times 0_x \cup 0_x \times \Gamma_2$$

and $WF(A_1 \otimes A_2) \cap N_\Delta = \varnothing$ means exactly $\Gamma_1 + \Gamma_2 \subset T^*(X)\setminus 0$ as we required. So A_1A_2 is defined, and

$$WF(A_1A_2) \subset \Delta^*(\Gamma_1 \times \Gamma_2 \cup \Gamma_1 \times 0_x \cup 0_x \times \Gamma_2) = \Gamma_1 + \Gamma_2 \cup \Gamma_1 \cup \Gamma_2. \qquad \text{Q.E.D.}$$

Let $K \in D'(X, Y)$. It defines a continuous map $K : C_0^\infty(Y) \to D'(X)$ by

$$\langle \psi, K\phi \rangle = K(\psi \otimes \phi) \quad \text{for } \phi \in C_0^\infty(Y),\ \psi \in C_0^\infty(X).$$

Theorem [1, **Theorem** 2.5.12]. *For any* $u \in C_0^\infty(Y)$, $WF(Ku) \subset \pi_*(WF(K))$ (*where* $\pi : X \times Y \to X$). $\pi_* WF(K)$ *is also denoted* $WF_x(K) = \{(x, \xi) : (x, \xi, y, 0) \in WF(K)$ *for some* $y \in Y\}$.

Proof. u gives $1 \otimes u \in D'(X \times Y)$, which is smooth. Therefore the product $K(1 \otimes u)$ is defined and $Ku = \pi_* K(1 \otimes u)$.

$$WF(K(1 \otimes u)) \subset WF(K) + WF(1 \otimes u) \cup WF(K) \cup WF(1 \otimes u)$$

but $1 \otimes u$ is smooth, so $WF(K(1 \otimes u)) \subset WF(K)$ which gives $WF(Ku) \subset \pi_* WF(K)$.

Theorem [1, Theorem 2.5.14]. *Let* $K \in D'(X \times Y)$, *and denote by*

$$WF'(K) = \{(x, \xi, y, -\eta) \in T^*(X) \times T^*(Y) : (x, \xi, y, \eta) \in WF(K)\}$$

a relation mapping sets in $T^*(Y)\backslash 0$ *to sets in* $T^*(X)\backslash 0$. *If* $WF_x(K) = \emptyset$, $u \in D'(Y)$ *and* $WF(u) \cap WF_Y'(K) = \emptyset$ (*where* $WF_Y'(K) = \{(y, \eta) : (x, 0, y, -\eta) \in WF(K)$ *for some* $x\}$) *then*

$$WF(Ku) \subset WF'(K)\, WF(u)$$

for $Ku \in D'(X)$ *defined by*

$$\langle \phi, Ku \rangle = \langle \phi \otimes 1, K(1 \otimes u) \rangle, \quad \phi \in C_0^\infty(x).$$

Proof. If everything is defined,

$$Ku = \pi_*(K(1 \otimes u)), \quad \pi : X \times Y \to X.$$

Since $WF_x(K) = \emptyset$, $K(1 \otimes u)$ is defined and

$$WF(K(1 \otimes u)) \subset WF(K) + 0_x \times WF(u) \cup WF(K) \cup 0_x \times WF(u).$$

For any $P, Q \subset T^*(X)$, $\pi_*(P \cup Q) = \pi_*(P) \cup \pi_*(Q)$. By our assumption $\pi_* WF(K) = \emptyset$, and $\pi_*(0_x \times WF(u)) = \emptyset$ always, so $WF(Ku) \subset \pi_*(WF(K) + 0_x \times WF(u)) = WF'(K)\, WF(u)$. Q.E.D.

Let $K_1 \in D'(X \times Y)$, $K_2 \in D'(Y \times Z)$ be properly supported (that is, $\pi_* K_j$ are well defined for the projections π on the components). Denote $\Gamma_j = WF(K_j)$.

Theorem [1, Theorem 2.5.15]. *If* $\Gamma_1 \times 0_z + 0_x \times \Gamma_2 \subset T^*(X \times Y \times Z)\backslash 0$, *then the composition* $K = K_1 \circ K_2 \in D'(X \times Z)$ *is well defined, and*

$$WF'(K_1 \circ K_2) \subset WF'(K_1) \circ WF'(K_2) \cup (WF_x(K_1) \times 0_z) \cup (0_x \times WF_z'(K_2)).$$

Proof. For smooth distributions of the form

$$K_1 = f(x) \otimes g_1(y), \quad K_2 = g_2(y) \otimes h(z),$$

the composition is defined as $f(x) \otimes h(z) \cdot \langle g_1, g_2 \rangle$. In other words,

$$(f \otimes g_1) \circ (g_2 \otimes h) = \pi_*((f \otimes g_1 \otimes 1)(1 \otimes g_2 \otimes h))$$

where $\pi : X \times Y \times Z \to X \times Z$.

In the continuous extension of this definition, we define, where possible, $K_1 \circ K_2 = \pi_*((K_1 \otimes 1_z)(1_x \otimes K_2))$ for the above π.

The restriction on defining products limits us to the case $0 \notin \Gamma_1 \times 0_z + 0_x \times \Gamma_2$. This holding, we get

$$WF(K_1 \circ K_2) \subset \pi_*(\Gamma_1 \times 0_z + 0_x \times \Gamma_2 \cup \Gamma_1 \times 0_z \cup 0_x \times \Gamma_2)$$

$$= WF'(K_1) \circ WF(K_2) \cup (WF_x(K_1) \times 0_z) \cup (0_x \times WF_z(K_2))$$

or

$$WF'(K_1 \circ K_2) \subset WF'(K_1) \circ WF'(K_2) \cup (WF_x(K_1) \times 0_z) \cup (0_x \times WF'_z(K_2)). \qquad \text{Q.E.D.}$$

Note that $0 \notin \Gamma_1 \times 0_z + 0_x \times \Gamma_2$ holds in particular when $WF'_Y(K_1) \cap WF_Y(K_2) = \varnothing$, which is the restriction in [1], however composition exists in less restrictive circumstances.

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